Domain wall solutions of KdV like equations with higher order nonlinearity

1986 J. Phys. A: Math. Gen. 19 L9

(http://iopscience.iop.org/0305-4470/19/1/003)

J. Phys. A: Math. Gen. 19 (1986) L9–L12. Printed in Great Britain

LETTER TO THE EDITOR

Domain wall solutions of KdV like equations with higher order nonlinearity

Bishwajyoti Dey
Institute of Physics, Bhubaneswar-751005, India

Received 21 August 1985

Abstract. We consider certain nonlinear partial differential equations which are Korteweg-de Vries (KdV) like equations with higher order nonlinearity. We show that these have got kink (domain wall) solutions for particular values of the coefficients of the nonlinear terms. The solutions are compared with the standard known solution of the $\lambda\phi^{2n}$ field theories. Some conservation laws for these system of equations are also given.

We consider four forms of nonlinear partial differential equations.

(i) The first form

$$u_t + a(1+bu)uu_x + \delta u_{xxx} = 0, \qquad a, \delta > 0 \tag{1a}$$

can be derived from the Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}\theta_x\theta_t + \tfrac{1}{6}a(1+\tfrac{1}{2}b\theta_x)\theta_x^3 - \tfrac{1}{2}\delta\psi^2 \tag{1b}$$

where $\theta_x = u$ and $\psi = \theta_{xx}$. This equation is like the combined KdV equation, which for $b=0$ reduces to the usual KdV equation (Novikov *et al* 1984). The subscripts denote partial derivatives.

(ii) The second form

$$[a(1+bu)u - \eta]u_x + \delta u_{xxx} = 0, \qquad \eta/a,\ \eta/\delta > 0 \tag{2a}$$

can be derived from

$$\mathscr{L} = [\tfrac{1}{6}a(1+\tfrac{1}{2}b\theta_x)\theta_x - \tfrac{1}{2}\eta]\theta_x^2 - \tfrac{1}{2}\delta\psi^2. \tag{2b}$$

This equation is comparable with the static case above (1*a*), except for an additional term ηu_x.

(iii) The third form

$$u_t + bu^2u_x - \delta u_{xxx} = 0, \qquad b, \delta > 0, \tag{3a}$$

is the modified KdV equation (Novikov *et al* 1984) with the exception of the sign of the highest derivative term. This equation can be derived from

$$\mathscr{L} = \tfrac{1}{2}\theta_x\theta_t + \tfrac{1}{12}b\theta_x^4 + \tfrac{1}{2}\delta\psi^2. \tag{3b}$$

(iv) The fourth form is

$$u_t + a(1+bu^2)u^2u_x + \delta u_{xxx} = 0, \qquad a, \delta > 0 \tag{4a}$$

0305-4470/86/010009+04$02.50

which for $b=0$ becomes the modified KdV equation. It can be derived from

$$\mathscr{L}=\tfrac{1}{2}\theta_x\theta_t+\tfrac{1}{6}a\theta_x^4(\tfrac{1}{2}+\tfrac{1}{5}b\theta_x^2)-\tfrac{1}{2}\delta\psi^2. \tag{4b}$$

In order to look for travelling solitary wave solutions we make the simple transformation

$$\xi=x-ct \tag{5}$$

where c is the propagation velocity of the solitary waves. With this transformation (1a) reduces to the stationary equation

$$-cu_\xi+a(1+bu)uu_\xi+\delta u_{\xi\xi\xi}=0, \tag{6a}$$

which, after integrating twice WRT ξ, can be rewritten as

$$-\tfrac{1}{2}cu^2+\tfrac{1}{6}au^3+\tfrac{1}{12}abu^4+\tfrac{1}{2}\delta u_\xi^2=k_1u+k_2 \tag{6b}$$

where k_1 and k_2 are constants of integration. Similarly (2a) reduces to

$$-\tfrac{1}{2}\eta u^2+\tfrac{1}{6}au^3+\tfrac{1}{12}abu^4+\tfrac{1}{2}\delta u_x^2=k_1u+k_2, \tag{7}$$

equation (3a) reduces to

$$\tfrac{1}{2}cu^2-\tfrac{1}{12}bu^4+\tfrac{1}{2}\delta u_\xi^2=k_1u+k_2 \tag{8}$$

and finally (4a) reduces to

$$-\tfrac{1}{2}cu^2+\tfrac{1}{12}au^4+\tfrac{1}{30}abu^6+\tfrac{1}{2}\delta u_\xi^2=k_1u+k_2. \tag{9}$$

Integration of (6b), (7), (8) and (9) gives, respectively, the domain wall (kink) solutions of (1a)-(4a). Thus the solutions of (1a) are given by

$$u(x,t)=(3c/a)\{1\pm\tanh[(c/\delta)^{1/2}\xi/2]\} \tag{10}$$

for $k_1=k_2=0$ and $b=-a/6c$. The $\pm$ sign corresponds to kink and antikink solutions, respectively. As ξ varies from $-\infty$ to $+\infty$, the kink and antikink solutions interpolate between 0 and $6c/a$ and between $6c/a$ and 0, respectively. These solutions resemble the kink and antikink solutions of $\lambda\phi^2(\varphi-1)^2$ field theory (Dey 1985).

The solutions of (2a) (static solutions) are given by

$$u(x)=\frac{3\eta}{a}\{1\pm\tanh[(\eta/\delta)^{1/2}x/2]\} \tag{11}$$

for $k_1=k_2=0$ and $b=-a/6\eta$. As in the above case the $\pm$ sign corresponds to the kink and antikink solutions respectively.

The kink and antikink ($\pm$sign) solution of (3a) are given by

$$u(x,t)=\pm(3c/b)^{1/2}\tanh[(c/2\delta)^{1/2}(\xi+k_3)] \tag{12}$$

for $k_1=0$, $k_2=3c^2/4b$ and k_3 a constant of integration. It can be noted that these solutions resemble the kink and antikink solutions of $\lambda\phi^4$ field theory (Khare 1979).

Finally the kink and antikink solutions of (4a) are given by

$$u(x,t)=\pm(6c/a)^{1/2}[1\pm\tanh(c/\delta)^{1/2}\xi]^{1/2} \tag{13}$$

for $k_1=k_2=0$ and $b=-5a/48c$. Out of these four solutions, the kink solutions are $u=(6c/a)^{1/2}\{1+\tanh[(c/\delta)^{1/2}\xi]\}^{1/2}$ and $u=-(6c/a)^{1/2}\{1-\tanh[(c/\delta)^{1/2}\xi]\}^{1/2}$, the remaining two are antikink solutions. These solutions resemble the kink and antikink solutions of $\lambda\phi^6$ field theory (Khare 1979).

Now we write some conservation laws for these types of equations. A conservation law associated with a KdV like equation is expressed by an equation of the form (Miura *et al* 1968)

$$T_t + X_x = 0 \tag{14}$$

where T the conserved density and $-X$, the flux of T, are functions of $u(x, t)$.

Considering (1a), we can immediately write down two conservation laws, the T and X values which are given by

$$T_1 = u \qquad X_1 = au^2(\tfrac{1}{2}+\tfrac{1}{3}bu) + \delta u_{xx} \tag{15}$$

and

$$T_2 = u^2 \qquad X_2 = \tfrac{1}{3}au^3 + \tfrac{1}{4}abu^4 + \delta uu_{xx} - \tfrac{1}{2}\delta u_x^2. \tag{16}$$

Similarly, we can write down two conservation laws for (4a), the T and X values for which are given by

$$T_1 = u \qquad X_1 = \tfrac{1}{3}au^3 + \tfrac{1}{5}abu^5 + \delta u_{xx} \tag{17}$$

and

$$T_2 = \tfrac{1}{2}u^2 \qquad X_2 = \tfrac{1}{4}au^4 + \tfrac{1}{6}abu^6 + \delta uu_{xx} - \tfrac{1}{2}\delta u_x^2. \tag{18}$$

Other conservation laws for (1a) and (4a) are not obvious and we are currently trying to find some of them (if there are any).

Equation (3a) is a more interesting case, because as for the modified KdV equation (Miura *et al* 1968) we can write many more conservation laws. The first four values of T and X are given below:

$$T_1 = u \qquad X_1 = \tfrac{1}{3}bu^3 - \delta u_{xx} \tag{19}$$

$$T_2 = \tfrac{1}{2}u^2 \qquad X_2 = \tfrac{1}{4}bu^4 - \delta uu_{xx} + \tfrac{1}{2}\delta u_x^2 \tag{20}$$

$$T_3 = \tfrac{1}{4}bu^4 + \tfrac{3}{2}\delta u_x^2 \qquad X_3 = \tfrac{1}{6}b^2u^6 - b\delta u^3u_{xx} + 3b\delta u^2u_x^2 - 3\delta^2u_xu_{xxx} + \tfrac{3}{2}\delta^2u_{xx}^2 \tag{21}$$

and finally

$$T_4 = \tfrac{1}{6}b^2u^6 + 5b\delta u^2u_x^2 + 3\delta^2u_{xx}^2$$

$$X_4 = \tfrac{1}{8}b^3u^8 - b^2\delta u^5u_{xx} - 10b\delta^2u^2u_xu_{xxx} + \tfrac{15}{2}b^2\delta u^4u_x^2 + 10b\delta^2uu_x^2u_{xx}$$
$$+ \tfrac{1}{2}b\delta^2u_x^4 + 8b\delta^2u^2u_{xx}^2 - 6\delta^3u_{xx}u_{xxxx} + 3\delta^3u_{xxx}^2. \tag{22}$$

The Hamiltonian nature of these system of equations is not obvious and needs further critical examination. Presently we are looking into this problem.

In conclusion we say that we have obtained exact domain wall (kink) solutions of KdV like nonlinear partial differential equations with higher order nonlinearity. The solutions are compared with the standard kink solutions of $\lambda\phi^{2n}$ theories. The Lagrangian density of each of these equations is written down. We have also written down a few conservation laws for these systems of equations. The results are important because until now no domain wall solutions of KdV like equations were known to exist.

The author would like to thank S N Behera, A Khare and S G Mishra for useful discussions and suggestions.

References

Dey B 1985 unpublished
Khare A 1979 *Lett Math. Phys.* **3** 475
Miura R M, Gardner C S and Kruskal M D 1968 *J. Math. Phys.* **9** 1204
Novikov S, Manakov S V, Pataevskii L P and Zakharov V E 1984 *Theory of Solitons* ed Revaz Gamkrelidge (New York: Consultants Bureau)